

# Radiative friction on an excited atom moving in vacuum

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## Abstract

It is known that, when an excited atom spontaneously emits one photon, two effects are produced. First, the atom's internal and external states are entangled with the states of the emitted photon. Second, the atom receives a momentum transferred from the photon. In this work, the dynamics of such an atom in vacuum is studied. Through a specific calculation, it is demonstrated that these effects cause the atom to experience, on average, a friction force opposite to its initial velocity. Properties of the force are also discussed.

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Because of atom-vacuum interaction, an excited atom is able to emit a photon spontaneously into any direction [1], and consequently has its internal and external states entangled with photonic states [2, 3]. Thus, even inside the vacuum, the atom's motion can never be a free motion. Already pointed out was that, in the presence of such photon-atom entanglement, the atom's position-momentum uncertainty can be temporarily lower than the lower bound specified by the Heisenberg uncertainty relation [3]. It is certainly desirable to know what other effects the atom-vacuum interaction can have on the atom's motion.

Note, the frequency of the emitted photon depends on the direction in which the photon is emitted according to the Doppler effect. In particular, when the photon is in the same direction as the atom's velocity, it must have a frequency larger than the frequency it has when emitted in the opposite direction. On the other hand, the photon emitted in any two directions perpendicular to the velocity must have the same frequency. Since a higher frequency means that a larger momentum is carried away by the photon, and that a larger momentum is transferred to the atom from the photon, the excited atom, when moving, must receive, on average, a nonzero momentum transferred from the photon, and should experience, in addition to the photon-atom entanglement, a radiative force, on average, opposite to its velocity. The present paper is devoted to the derivation of the radiative force  $\vec{F}_r$ . Since the radiative force is opposite to the atom's velocity, it is addressed as radiative friction in the paper. In the literature, radiative forces on an atom are often discussed when the atom is exposed to laser beams [1, 4], and are seldom analyzed when the atom is simply in the vacuum. Unlike the laser beams, the vacuum has far more electromagnetic modes to be considered. It will become evident in the following discussion that  $\vec{F}_r$  also depends critically on the photon-atom entanglement.

Denote the momentum operator, position operator, and mass of the excited atom as  $\vec{P}$ ,  $\vec{R}$ , and  $m$  respectively. For simplicity, the atom is assumed to have two energy states: an excited state  $|E\rangle$  with energy  $\hbar\omega_E$  and the ground state  $|G\rangle$  with energy  $\hbar\omega_G$ . The difference between  $\omega_E$  and  $\omega_G$  is known as the atomic transition frequency  $\omega_0 \equiv \omega_E - \omega_G$ . The interaction between the vacuum modes and atom is assumed, as usual, to be through the atom's electric dipole moment  $\vec{\mu}$  and is described by an operator  $V_I$  defined as

$$V_I = \sum_{\alpha} \left( \vec{\mu}_{GE} \cdot \vec{g}_{\alpha} e^{-i\vec{k}_{\alpha} \cdot \vec{R}} a_{\alpha}^{\dagger} |G\rangle \langle E| + \vec{\mu}_{EG} \cdot \vec{g}_{\alpha}^{*} e^{i\vec{k}_{\alpha} \cdot \vec{R}} a_{\alpha} |E\rangle \langle G| \right), \quad (1)$$

where  $\vec{\mu}_{GE}$  is the matrix element of  $\vec{\mu}$  between  $|G\rangle$  and  $|E\rangle$ , and  $\vec{\mu}_{EG}$  the complex conjugate

of  $\vec{\mu}_{GE}$ . The frequency and amplitude (containing a polarization unit vector  $\vec{\epsilon}_\alpha$ ) of mode  $\alpha$  are represented by  $\omega_\alpha$  and  $\vec{g}_\alpha = i\sqrt{2\pi\hbar\omega_0^2/(L^3\omega_\alpha)}\vec{\epsilon}_\alpha$  respectively. The quantization volume is  $L^3$ . Also used in  $V_I$  are  $\vec{k}_\alpha$ , the wave vector of mode  $\alpha$  ( $|\vec{k}_\alpha| = \omega_\alpha/c$ ), and  $a_\alpha^\dagger$  ( $a_\alpha$ ), the creation (annihilation) operator for the same mode. Throughout the paper,  $c$  is the speed of light in the vacuum, and  $*$  denotes the complex conjugate of a quantity. Still for simplicity,  $V_I$  is treated under the rotating-wave approximation [5]. The Hamiltonian  $H$  of the atom-vacuum system is constructed by adding to  $V_I$  the unperturbed Hamiltonian  $H_0 = P^2/(2m) + \hbar\omega_E|E\rangle\langle E| + \hbar\omega_G|G\rangle\langle G| + \sum_\alpha \hbar\omega_\alpha a_\alpha^\dagger a_\alpha$ :

$$H = H_0 + V_I. \quad (2)$$

Since it is constant and unimportant to the atom-vacuum system's evolution, the zero point energy of the vacuum is ignored in  $H$ . Also ignored is the Röntgen interaction, because this interaction [6, 7] is roughly of the order of  $vc^{-1}V_I$  (where  $v$  is the atom's speed), much weak compared with  $V_I$  in the present nonrelativistic analysis. For example, in a typical experiment on spontaneous emission [8],  $v$  is merely of the order of  $10^3 m/s$ .

The atom's initial external state  $|\psi(0)\rangle = \int d\vec{p}_0 f(\vec{p}_0)|\vec{p}_0\rangle$  is expressed in terms of the eigenstates  $|\vec{p}_0\rangle$  of the momentum operator. To mimic the initial condition, assumed without loss of generality, that the atom moves along the positive direction  $\hat{x}$  of the  $x$ -axis of a coordinate system stationary in the vacuum, the eigenvalues  $\vec{p}_0$  of these states are assumed to be all along  $\hat{x}$ . Since the atom is in the excited state  $|E\rangle$ , and no photons are present  $|0\rangle$ , evolution of the atom-vacuum system must start from such an initial state  $|\phi(0)\rangle = |\psi(0)\rangle \otimes |E\rangle \otimes |0\rangle$ . At time  $t$ , the state of the system  $|\phi(t)\rangle$  consequently becomes

$$|\phi(t)\rangle = e^{-\frac{iHt}{\hbar}}|\phi(0)\rangle = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dq \frac{e^{-iqt/\hbar}}{q - H} |\phi(0)\rangle. \quad (3)$$

As, for example, in the discussion of the Ehrenfest theorem [9] and the atomic motion in the laser beams [1, 4], the radiative friction  $\vec{F}_r$  is defined to be proportional to the second-order time derivative of the expectation value of the position operator  $\vec{R}$  taken with respect to state  $|\phi(t)\rangle$ :

$$\vec{F}_r = m \frac{d^2}{dt^2} \langle \phi(t) | \vec{R} | \phi(t) \rangle. \quad (4)$$

Consider the  $x$ -component of the force  $\vec{F}_r$ . From Eq. (3), the time-dependent expectation value of  $R_x$  (the  $x$ -component of  $\vec{R}$ ),

$$\langle R_x \rangle(t) = \langle \phi(t) | R_x | \phi(t) \rangle = \langle \phi(0) | e^{\frac{iHt}{\hbar}} R_x e^{-\frac{iHt}{\hbar}} | \phi(0) \rangle, \quad (5)$$

and its first-order derivative,

$$\begin{aligned}\frac{d}{dt}\langle R_x \rangle(t) &= \frac{i}{\hbar} \langle \phi(0) | e^{\frac{iHt}{\hbar}} [H, R_x] e^{-\frac{iHt}{\hbar}} | \phi(0) \rangle \\ &= \frac{1}{m} \langle \phi(0) | e^{\frac{iHt}{\hbar}} P_x e^{-\frac{iHt}{\hbar}} | \phi(0) \rangle,\end{aligned}\quad (6)$$

are first obtained. In Eq. (6),  $P_x$  is the  $x$ -component of the momentum operator  $\vec{P}$ . From the preceding equation, it is then a straightforward matter to find the second-order time derivative of  $\langle R_x \rangle(t)$ :

$$\begin{aligned}\frac{d^2}{dt^2}\langle R_x \rangle(t) &= \frac{i}{m} \langle \phi(t) | \sum_{\alpha} (\vec{\mu}_{GE} \cdot \vec{g}_{\alpha}) a_{\alpha}^{\dagger} | G \rangle \langle E | k_{\alpha x} e^{-i\vec{k}_{\alpha} \cdot \vec{R}} | \phi(t) \rangle \\ &\quad - \frac{i}{m} \langle \phi(t) | \sum_{\alpha} (\vec{\mu}_{EG} \cdot \vec{g}_{\alpha}^*) a_{\alpha} | E \rangle \langle G | k_{\alpha x} e^{i\vec{k}_{\alpha} \cdot \vec{R}} | \phi(t) \rangle,\end{aligned}\quad (7)$$

where  $k_{\alpha x}$ , as the  $x$ -component of the mode vector  $\vec{k}_{\alpha}$ , illustrates the momentum exchange between the photon and atom when the photon is created or annihilated.

The evolution of the atom-vacuum system is accompanied by two simultaneous processes: spontaneous emission and associated atomic recoil from the emitted photon. The net result of these processes is that, as noted before, the atomic and photonic states are entangled [3]:

$$\begin{aligned}|\phi(t)\rangle &= -\frac{1}{2\pi i} \int dq e^{-iqt/\hbar} \int d\vec{p}_0 \frac{f(\vec{p}_0)}{q - p_0^2/(2m) - \hbar\omega_E - B} |\vec{p}_0\rangle \otimes |E\rangle \otimes |0\rangle \\ &\quad - \frac{1}{2\pi i} \int dq e^{-iqt/\hbar} \int d\vec{p}_0 \frac{f(\vec{p}_0)}{q - p_0^2/(2m) - \hbar\omega_E - B} \\ &\quad \times \sum_{\alpha} \frac{(\vec{\mu}_{GE} \cdot \vec{g}_{\alpha}) e^{-i\vec{k}_{\alpha} \cdot \vec{R}} |\vec{p}_0\rangle \otimes |G\rangle \otimes |1_{\alpha}\rangle}{q - (\vec{p}_0 - \hbar\vec{k}_{\alpha})^2/(2m) - \hbar\omega_G - \hbar\omega_{\alpha}},\end{aligned}\quad (8)$$

where  $p_0 = |\vec{p}_0|$ , and

$$\begin{aligned}B &\simeq -\frac{\Gamma_0 \hbar}{2\omega_0 \pi} \left[ \Omega + \omega_0 \ln \left( \frac{\Omega - \omega_0}{\omega_0} \right) \right] - i \frac{\Gamma_0 \hbar}{2} \\ &\equiv B_r + iB_i.\end{aligned}\quad (9)$$

In the expression of  $B$ , the quantity  $\Gamma_0 = 4 |\vec{\mu}_{GE}|^2 \omega_0^3 / (3\hbar c^3)$  is the spontaneous emission rate of a stationary atom in the vacuum, and  $\Omega$  a cut-off frequency needed to make the nonrelativistic Hamiltonian  $H$  applicable in the present discussion [10]. Note, in Eq. (8),  $(\vec{p}_0 - \hbar\vec{k}_{\alpha})^2/(2m)$  is the atom's kinetic energy after the atom recoils from the photon emitted into mode  $|1_{\alpha}\rangle$ .

Substitute Eq. (8) into Eq. (7) to get

$$\frac{d^2}{dt^2}\langle R_x \rangle(t) = -\frac{\omega_0^2 |\vec{\mu}_{GE}|^2}{3m^2 c^5} (\omega_0 + B_r/\hbar)^2 e^{-\Gamma_0 t} \int d\vec{p}_0 |f(\vec{p}_0)|^2 p_0. \quad (10)$$

In the derivation of Eq. (10),  $\vec{\mu}_{GE}$  is averaged over its orientation to conform to the fact that the orientation is usually unknown. Since the dependence of the atom's spontaneous emission on the atom's speed is weak [11], the contribution from the Doppler effect is ignored in the frequency of the emitted photon ( $\omega_0 + B_r/\hbar$ ). Also used in Eq. (10) is the mode-continuum approximation; see, for example, Ref. [12]. A comparison of Eqs. (7) and (8) shows that the derivative in Eq. (10) can never survive unless the photonic and atomic states are entangled as in Eq. (8).

Similarly, it is found that the second-order time derivatives of  $\langle R_y \rangle(t)$  and  $\langle R_z \rangle(t)$  both vanish, where  $R_y$  and  $R_z$  are the  $y$ - and  $z$ -components of  $\vec{R}$  respectively. Thus, the radiation friction  $\vec{F}_r$  on the atom must be

$$\begin{aligned}\vec{F}_r &= -\frac{\omega_0^2 |\vec{\mu}_{GE}|^2}{3mc^5} (\omega_0 + B_r/\hbar)^2 e^{-\Gamma_0 t} \int d\vec{p}_0 |f(\vec{p}_0)|^2 \vec{p}_0 \\ &= -\frac{(\omega_0 + B_r/\hbar)^2 \hbar \Gamma_0}{4m\omega_0 c^2} e^{-\Gamma_0 t} \int d\vec{p}_0 |f(\vec{p}_0)|^2 \vec{p}_0.\end{aligned}\quad (11)$$

Three properties of  $\vec{F}_r$  are recognized. First, the magnitude of  $\vec{F}_r$  decays with time exponentially. This observation is understandable, because the force is only present when the atom and photon interact during spontaneous emission, and spontaneous emission is largely an exponential process. For a discussion of spontaneous emission beyond the rotating-wave approximation, see, for example, Ref. [12]. Second, the force is proportional to the average (initial) momentum of the atom, a character of the average Langevin force if the atom's motion is viewed as a Brownian motion [13]. Physically, the average momentum determines the difference between the photonic frequencies when the photon is emitted in or opposite to the direction of the atomic velocity, and, thus, should also determine the strength of the friction force. A stationary atom, whose momentum is zero, certainly does not experience the friction force  $\vec{F}_r$ . Finally, since it is proportional to the initial momentum of the atom, the force  $\vec{F}_r$  must depend on the coordinate system in which it is observed. As Eq. (7) shows, such dependence comes from the fact that  $\frac{d^2}{dt^2} \langle R_x \rangle(t)$  is related to the wave vector  $\vec{k}_\alpha$  of the photon, which, when combined with  $i\omega_\alpha/c$ , is a 4-vector, and is different in different systems. The friction force is not an invariant under the Lorentz transformation.

In conclusion, it is demonstrated that the motion of an excited atom in the vacuum is subject to a friction force. The force comes not only from the photon-atom entanglement

but also from the momentum transferred from the emitted photon to the atom.

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